Spinors,Bott periodicity and Dimension of Physics theories

Jiashen Chen

USTC

5.5.2025

Summary

One particularly interesting thing when we learn spinors, which are the basic components of the SUSY algebra, is the special properties of its relations with spacetime dimension. For example,only particular dimensions can admit Majorana spinors and Weyl decomposition of chirality can only be performed in even dimensions.

Introduction

This relation is not just a mathematical result, and it has physical meanings which are also interesting.The Majorana-Weyl condition only admits spacetime dimension that

$$d = 2mod8 \tag{1}$$

that gives $d = 10$ satisfies this condition, and TypeIIB superstring lives here has Majorana fermions.We already know that $d = 10$ comes from anomaly cancelling, so this is more interesting than a coincidence.

Spinors and Clifford Algebra

As a simple conclusion, we already know spinors are representations of Spin group.

Spinors

$$Spin(p,q) \to Hom_K(W,W), K = \mathbb{R}, \mathbb{C}, \mathbb{H} \tag{2}$$

where W is a K vector field

It is well known that Spin group is a double cover of the special orthogonal group, but we can also understand Spin group as a subgroup of Clifford algebra.

Spinors and Clifford Algebra

Clifford algebra are constructed directly from a vector field, and inversible and normal part with even grading is defined as Spin group.

Clifford Algebra

for V is a vector space and Q is a quadraic form,$T = \oplus_{n\geq 0} V^{\otimes n}$ is a tensor algebra. Then

$$Cl(V,Q) := T/\{x^2 = -Q(x), x \in V\} \tag{3}$$

is a Clifford algebra, denote $Cl^{\times}(V,Q)$ as the muliplication group generated by all inversible elements.

Of course we have the simple decomposition of $T = T^{even} \oplus T^{odd}$ which gives $Cl(V,Q) = Cl^0 V, Q \oplus Cl^1(V,Q)$.

Pin and Spin

$$Pin(V,Q) := \{v_1, ..., v_r \in Cl^{\times}(V,Q) | v_i \in V, Q(v_i) = \pm 1\} \tag{4}$$

$$Spin(V,Q) := Pin(V,Q) \cap Cl^0(V,Q) \tag{5}$$

This shows that the dimension of spinor is related to the dimension of representation of Clifford algebra. And studying the latter one can give us all results of the former one.

Bott Periodicity in algebra

Bott Periodicity

Clifford algebra has a isomorphic relation that states

$$Cl_{p+4,q} \simeq Cl_{p,q+4} \tag{6}$$

$$Cl_{p+8,q} \simeq Cl_{p,q}(16) \tag{7}$$

Here we denote the Clifford algebra generated by a pesudo-Riemannian metric with signature (p, q) as $Cl_{p,q}$, and a D dimensional matrix space generated by V as $V(D)$.
This gives the periodicity of representation of these algebras, and generally the periodicity states that

$$\tilde{D} = p - q \quad mod 8 \tag{8}$$

generates the same representation.

Bott Periodicity in algebra

n	1	2	3	4	5	6	7	8
$Cl_{n,0}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	$\mathbb{R}(16)$
$Cl_{0,n}$	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$
$Cl_{n-1,1}$	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$
$Cl_{1,n-1}$	$\mathbb{C}$	$\mathbb{R}(2)$	$\mathbb{R}(2)\oplus\mathbb{R}(2)$	$\mathbb{R}(4)$	$\mathbb{C}(4)$	$\mathbb{H}(4)$	$\mathbb{H}(4)\oplus\mathbb{H}(4)$	$\mathbb{H}(8)$
$\mathbb{C}l_n$	$\mathbb{C}\oplus\mathbb{C}$	$\mathbb{C}(2)$	$\mathbb{C}(2)\oplus\mathbb{C}(2)$	$\mathbb{C}(4)$	$\mathbb{C}(4)\oplus\mathbb{C}(4)$	$\mathbb{C}(8)$	$\mathbb{C}(8)\oplus\mathbb{C}(8)$	$\mathbb{C}(16)$

Bott Periodicity in topology

It is clear to us that the Orthogonal group $O(n)$ is a Lie group, and considering that $M(n, \mathbb{R})$ has a standard topology as a Euclidean space, $O(n)$ can be equipped with a subspace topology and also it is compact from Heine-Borel Thm.
From here, we are able to define

direct limit space of $O(n)$

As a topological space,$U \in O(\infty)$ is open if and only if U is open in $O(n)$ for any n.

and another way to see this is from the natural insertion of $O(n) \in O(n+1)$, and

$$O(\infty) = \lim_{n \to \infty} O(n) \tag{9}$$

is a formal definition.

Bott Periodicity in topology

Bott Periodicity

$$\pi_n(O(\infty)) \simeq \pi_{n+8}(O(\infty)) \tag{10}$$

more explicitly, we have
$\pi_0(O(\infty)) \simeq \mathbb{Z}_2$,$\pi_1(O(\infty)) \simeq \mathbb{Z}_2$,$\pi_2(O(\infty)) \simeq 0$,$\pi_3(O(\infty)) \simeq \mathbb{Z}$,$\pi_4(O(\infty)) \simeq 0$,$\pi_5(O(\infty)) \simeq 0$,$\pi_6(O(\infty)) \simeq 0$,$\pi_7(O(\infty)) \simeq \mathbb{Z}$

A natural question is how can we associate these two periodicity both with period of 8 together, and that requires us to know some definition of Classifying spaces.

Classifying spaces

isomorphism on principal G-bundles

A principal G bundle is a bundle that has a G action on its full space and preserves its bundle.For $f : P \to P'$ are two full spaces of principal G bundles and the graph is commutative, then the two bundles are isomorphic.

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "\pi"'] & P' \arrow[d, "\pi'"] \\ M \arrow[r, equal] & M \end{tikzcd}$$

Classfying spaces

For any spaces X, the homotopy class of continous map $f : x \to BG$ isomorphic to the isomorphic class of G principal bundle on X

Classifying spaces

The reason that BG classifies something is when we look at simple examples like $BO(n)$, because $O(n)$ bundles are real vector bundles, so $BO(n)$ classifies all real vector bundles on X.

Stable classifying space

$$BO(\infty) := \lim_{n\to\infty} BO(n) \tag{11}$$

which we often denote as BO.

We need to connect BO and O to push our expect result to a more explicit area

Universal bundle and Universal spaces

Universal bundle is a special G principal bundle which base space is the classifying space $\pi : EG \to BG$ and has free action $BG = EG/G$ as well as contractable to all orders $\pi_k(EG) = 0$.

Classifying spaces

From its free action, a direct result is a fibration

$$G \to EG \to BG \tag{12}$$

or a exact sequence of its homotopy group, considering homotopy group preserves its fibration

$$\pi_k(G) \to \pi_k(EG) \to \pi_k(BG) \tag{13}$$

and considering the snake lemma, we have

$$... \to \pi_{k+1}(BG) \to \pi_k(G) \to \pi_k(EG) \to \pi_k(G) \to \pi_k(BG) \to ... \tag{14}$$

use the all order contractable property in EG's definition, we have

$$\pi_{k+1}(BG) \simeq \pi_k(G) \tag{15}$$

Therefore we know that $\pi_k(BO)$ also has a periodicity of period 8 from the Bott periodicity.

K groups

Stable equivalence and K group

For vector bundles $E, F \to X$, $E \sim F$ when $\exists k$ that $E \oplus \mathbb{R}^k \simeq F \oplus \mathbb{R}^k$,which is called stable equivalence. And

$$KO(x) := \{[E] - [F] | [E], [F] \in C\} \tag{16}$$

where C is the equivalence class of stable equivalence between real vector bundles on X, where this construction is a special case of Grothendieck group.

The core lemma we will use is the relation between stable classifying space and K group,which is also a definition to K group in some places,

$$KO(X) \simeq [X, BO \times \mathbb{Z}] \tag{17}$$

where $[X, Y]$ denotes all homotopy equivalent class of $f : X \to Y$.

A short proof of $KO(X) \simeq [X, BO \times \mathbb{Z}]$

(1)We define

$$\Phi(E) : X \to BO \times \mathbb{Z}, x \to ([E_x], dimE - dimE_0) \tag{18}$$

for $E \to X$ where $[E_x] \in BO$ is the fiber at $x \in X$ and E_0 is a trivial bundle.It is easy to see if E and F are stable equivalent, $\Phi(E)$ and $\Phi(F)$ differs in a simple translation, which implies their homotopic equivalence $\Phi(E) \sim \Phi(F)$.

(2)If $\Phi(E) = \Phi(F)$, then there is a homotopy map $H : X \times [0, 1] \to BO \times \mathbb{Z}$ and deforms $E_0 = E$ to $E_1 = F$, which means E is stable equivalent to F.

(3)Also any $f : X \to BO \times \mathbb{Z}, x \to ([E], n(x))$ equals to find a trivial bundle $\mathbb{R}^{n(x)} \to X$, means Φ is surjection.

(4)Therefore Φ is a 1to1 map from stable equivalent classes to $[X, BO \times \mathbb{Z}]$.

connecting K group and Classifying space

loop space

$$\Omega X = \{f : [0,1] \to X | f(0) = f(1) = x_0\} \tag{19}$$

with $U(K,V) = \{f \in \Omega X | f(K) \subseteq V\}$ as its topology

and

K group with degrees

$$KO^{-n}(X) = [X, \Omega^n(BO \times \mathbb{Z})] \tag{20}$$

from Bott periodicity of BO: $\pi_{k+8}(BO) = \pi_k(BO)$ then

$$\Omega^8(BO \times \mathbb{Z}) \simeq BO \times \mathbb{Z} \tag{21}$$

implies

$$KO^{-n}(X) \simeq KO^{-n-8}(X) \tag{22}$$

which is the Bott periodicity of K groups.

A short summary

$$
\begin{tikzcd}
Cl_{p+8,q} \simeq Cl_{p,q} \arrow[r, dash, "?"] & KO^{-n}(X) \simeq KO^{-n-8}(X) \arrow[d, leftrightarrow, "KO(X) \simeq {[X, BO \times \mathbb{Z}]}"] \\
& \Omega^8(BO \times \mathbb{Z}) \simeq BO \times \mathbb{Z} \arrow[d, leftrightarrow] \\
& \pi_{k+8}(BO) \simeq \pi_k(BO) \arrow[d, leftrightarrow, "\pi_{k+1}(BO) \simeq \pi_k(O)"] \\
& \pi_{k+8}(O) \simeq \pi_k(O)
\end{tikzcd}
$$

First we define how a vector bundle admits Clifford algebra structure

Vector bundle with Clifford structure

If a vector bundle E's every bundle E_x is a module of Clifford algebra and every transition function commutes with Clifford action, then E is equipped with Cl_n structure.

Atiyah-Bott-Shapiro Theorem

$$KO^{-n}(X) \simeq \frac{\{E \quad with \quad Cl_n\}}{stable \quad equivalence} \tag{23}$$

where E is a real vector bundle on X.

this shows the periodicity of BO corresponds to $Cl_{n+8} \simeq Cl_n(16)$.

A short remark

The thm is very hard to prove and one can refer to the original paper of Atiyah-Bott-Shapiro, one thing I'd like to add is this implies while classifying all real vector bundles, noncommutative naturally occurs and is exposed by the vector bundle with Clifford structure.

One understanding in physics is that general theory gives objects whose charges have value in K group but not only in deRham cohomology groups, it comes to the RR charge of D brane in string theory. And noncommutative geometry should naturally occur from these objects, that this the so called string field theory.

Another understanding comes from view of condensed matter theory, one can refers to 0503006 for more.

Why $D = 1, 3, 7$?

vector product

Only spaces that $D = 0, 1, 3, 7$ can define a vector product.

Hurwitz Theorem

Any normed division algebra with a unit element can only be $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$

These directly comes from

Parallelizability Theorem

Only S^n that $n = 0, 1, 3, 7$ can find a trivial tangent bundle on them.

And this theorem is proved by solving a discrete equation comes from Bott periodicity, one can refers to Adams J F. Vector fields on spheres[J]. Annals of Mathematics, 1962, 75(3): 603-632.

Or directly to Hurwitz

n	1	2	3	4	5	6	7	8
$Cl_{n,0}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$
$S_{n,0}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}_\pm$	$\mathbb{H}^2$	$\mathbb{C}^4$	$\mathbb{R}^8$	$\mathbb{R}^8_\pm$	$\mathbb{R}^{16}$
Irreducible $\mathbb{R}$-spinors	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}_\pm$	$\mathbb{H}^2$	$\mathbb{C}^4$	$\mathbb{R}^8$	$\mathbb{R}^8_\pm$

Using $Cl_{n-1,0} = Cl^0_{n,0}$, we find

- $n \equiv 3,5,6,7 \mod 8$. S is irreducible, quaternion for $n = 3,5$, complex for $n = 6$, real for $n = 7$.
- $n \equiv 1 \mod 8$. S is a direct sum of two equivalent irreducible $\mathbb{R}$-representations.
- $n \equiv 2 \mod 8$. S is a direct sum of two equivalent irreducible $\mathbb{C}$-representations.
- $n \equiv 4 \mod 8$. S is a direct sum of two inequivalent irreducible $\mathbb{H}$-representations.
- $n \equiv 8 \mod 8$. S is a direct sum of two inequivalent irreducible $\mathbb{R}$-representations.

Another route is to derive Hurwitz theorem from CLifford Classification, and Hurwitz can derive the Parallelizability Theorem simply.

Example 3.16 (Minkowski space). The real and complex representations for Minkowski spaces are summarized as follows.

n	1	2	3	4	5	6	7	8
$Cl_{n-1,1}$	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$
$S_{n-1,1}$	$\mathbb{R}_\pm$	$\mathbb{R}^2$	$\mathbb{C}^2$	$\mathbb{H}^2$	$\mathbb{H}^2_\pm$	$\mathbb{H}^4$	$\mathbb{C}^8$	$\mathbb{R}^{16}$
Irred $\mathbb{R}$-spinor	$\mathbb{R}$ (M)	$\mathbb{R}_\pm$ (MW)	$\mathbb{R}^2$ (M)	$\mathbb{C}^2$ (W)	$\mathbb{H}^2$ (SM)	$\mathbb{H}^2_\pm$ (SMW)	$\mathbb{H}^4$ (SM)	$\mathbb{C}^8$ (W)
Cl_n	$\mathbb{C}\oplus\mathbb{C}$	$\mathbb{C}(2)$	$\mathbb{C}(2)\oplus\mathbb{C}(2)$	$\mathbb{C}(4)$	$\mathbb{C}(4)\oplus\mathbb{C}(4)$	$\mathbb{C}(8)$	$\mathbb{C}(8)\oplus\mathbb{C}(8)$	$\mathbb{C}(16)$
S_n	$\mathbb{C}$	$\mathbb{C}^2$	$\mathbb{C}^2$	$\mathbb{C}^4$	$\mathbb{C}^4$	$\mathbb{C}^8$	$\mathbb{C}^8$	$\mathbb{C}^{16}$
Irred $\mathbb{C}$-spinors	$\mathbb{C}$	$\mathbb{C}_\pm$	$\mathbb{C}^2$	$\mathbb{C}^2_\pm$	$\mathbb{C}^4$	$\mathbb{C}^4_\pm$	$\mathbb{C}^8$	$\mathbb{C}^8_\pm$

(1) $n \equiv 2 \mod 8$. The two chiral irreducible $\mathbb{R}$-spinors are **Majorana-Weyl** (MW) spinors.

(2) $n \equiv 6 \mod 8$. The two chiral irreducible $\mathbb{R}$-spinors are **Symplectic-Majorana-Weyl** (SMW) spinor.

(3) $n \equiv 4, 8 \mod 8$. The irreducible chiral $\mathbb{C}$-spinors are **Weyl** spinors. They give rise to equivalent $\mathbb{R}$-spinors which are complex conjugate of each other.

(4) $n \equiv 1, 3 \mod 8$. The irreducible $\mathbb{R}$-spinors are **Majorana** spinors.

(5) $n \equiv 5, 7 \mod 8$. The irreducible $\mathbb{R}$-spinors are **Symplectic-Majorana** spinors.

Relations to Spinors:A example of $Cl_{1,1}$

We have the quadraic form of

$$Q = -x_1^2 + x_2^2 \tag{24}$$

here,and we can denote as

$$Q = y^2 - x^2 \tag{25}$$

which gives relation

$$yx = -xy, x^2 = -y^2 = 1 \tag{26}$$

by our choice of sign, and a simple representataion is

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, xy = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which is a base of$\mathbb{R}(2)$

Relation to Spinors in physics

In physics, expecially SUSY gauge field theory, we can count the maximum number of SUSY by counting the real components of matter field(which is the spinors) as

$$N * Components_{real} \leq 32 \tag{27}$$

where the bound is given by a series of no go theorems which prevent elementary particles that has spin bigger than 2 exists.

This and the graph before strictly limits the dimension of physics theories, odd dimension cannot have good theory becuase of the absence of Weyl decomposition and when even dimension comes to 6 it is very different from lower ones because of the absence of Majorana spinors.

Summary

We have a brief journey from spinors to topology and K theory and back to physics, the magnificent connection between periodicity in different areas not only gives us a special view of constants that mysteriously occur in theorems,but also let us have a deeper view of noncommutative properties and topology in fundamental physics.